Published by Institute of Physics Publishing for SISSA
Received: *September 8, 2005*
Accepted: *November 14, 2005*
Published: *December 1, 2005*

SUSY anomalies break $\mathcal{N}=2$ to $\mathcal{N}=1$: the supersphere and the fuzzy supersphere

Aiyalam P. Balachandran, Aleksandr Pinzul and Babar Qureshi
Physics Department, Syracuse University
Syracuse, NY, 13244-1130, U.S.A.
E-mail: bal@phy.syr.edu, apinzul@physics.syr.edu, bqureshi@physics.syr.edu

Abstract: The $\mathcal{N}=1$ SUSY on S^2 and its fuzzy finite-dimensional matrix version (see [1, 2] and references therein) are known. The latter regulates quantum field theories, and seems suitable for numerical work and capable of higher dimensional generalizations. In this paper, we study their instanton sectors. They are SUSY generalizations of U(1) bundles on S^2 and their fuzzy versions, and can be characterized by $k \in \mathbb{Z}$, the SUSY Chern numbers. In the no-instanton sector ($k=0$), $\mathcal{N}=2$ SUSY can be chirally realized, the 3 new $\mathcal{N}=2$ generators anticommuting with the "Dirac" operator defining the free action. If $k \neq 0$, the Dirac operator has zero modes which form an $\mathcal{N}=1$ supermultiplet and an atypical representation of $\mathcal{N}=2$ SUSY. They break the chiral SUSY generators by the Fujikawa mechanism [3−5]. We have not found this mechanism for SUSY breakdown in the literature. All these phenomena occur also on the supersphere SUSY, the graded commutative limit of the fuzzy model. We plan to discuss that as well in a later work.

Keywords: Non-Commutative Geometry, Supersymmetry Breaking, Field Theories in Lower Dimensions.

Contents

JHEP12(2005)002

1. Overview

In this section, we give an overview of fuzzy SUSY as full details can be found elsewhere [6]. In later sections, we will explain all the requisite details to develop instanton theory.

1.1 The fuzzy sphere

We recall that the fuzzy sphere $S_F^2(n)$ is the $(n+1) \times (n+1)$ matrix algebra $Mat(n+1)$. It can be realized as linear operators on $\mathcal{H}^{n+1}$ with the orthonormal basis vectors

$$\frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} |0\rangle \,, \qquad n_1 + n_2 = n \,, \tag{1.1}$$

where a_i, $a_i^\dagger$ are bosonic oscillators. The vectors (1.1) span a subspace of the Fock space with fixed particle number n:

$$N := \sum_i a_i^\dagger a_i \,, \qquad N|_{\mathcal{H}^{n+1}} = n \,. \tag{1.2}$$

In this representation, the elements of $S_F^2(n)$ are the linear operators

$$\sum_{i,j} c_{i,j}^m (a_i^\dagger)^m (a_j)^m \,, \qquad c_{i,j}^m \in \mathbb{C}\,,$$

restricted to the subspace $\mathcal{H}^{n+1}$.

The group SU(2) acts on $\mathcal{H}^{n+1}$ and hence on $S_F^2(n)$ by its spin $\frac{n}{2}$ unitary irreducible representation. The angular momentum generators are

$$L_i = a^\dagger \frac{\sigma_i}{2} a\,, \qquad \sigma_i \text{ are Pauli matrices}\,. \tag{1.3}$$

1.2 SUSY

The $\mathcal{N} = 1$ SUSY version of SU(2) is OSp(2,1). It has the graded Lie algebra $osp(2,1)$. Its generators (basis) can be written using oscillators if we introduce one additional fermionic oscillator b and its adjoint $b^\dagger$. They commute with a_i, $a_j^\dagger$. Then the $osp(2,1)$ generators are

$$\begin{aligned} \Lambda_i &= a^\dagger \frac{\sigma_i}{2} a\,, \qquad \Lambda_4 = -\frac{1}{2}(a_1^\dagger b + b^\dagger a_2)\,, \\ \Lambda_5 &= \frac{1}{2}(-a_2^\dagger b + b^\dagger a_1)\,, \qquad \sigma_i = \text{Pauli matrices}\,. \end{aligned} \tag{1.4}$$

The $\mathcal{N} = 2$ SUSY version of SU(2) is OSp(2,2). It has the graded Lie algebra $osp(2,2)$. Its basis consists of the $osp(2,1)$ generators and three additional generators

$$\begin{aligned} \Lambda_{4'} &\equiv \Lambda_6 = \frac{1}{2}(a_1^\dagger b - b^\dagger a_2)\,, \qquad \Lambda_{5'} \equiv \Lambda_7 = \frac{1}{2}(a_2^\dagger b + b^\dagger a_1)\,, \\ \Lambda_8 &= a^\dagger a + 2b^\dagger b\,. \end{aligned} \tag{1.5}$$

If $\{\cdot,\cdot\}$ denotes the anticommutator, $osp(2,2)$ has the defining relations

$$\begin{aligned} &[\Lambda_i, \Lambda_j] = i\varepsilon_{ijk}\Lambda_k\,, \qquad [\Lambda_i, \Lambda_\alpha] = \frac{1}{2}\Lambda_\beta(\sigma_i)_{\beta\alpha}\,, \qquad \{\Lambda_\alpha, \Lambda_\beta\} = \frac{1}{2}(\varepsilon\sigma_i)_{\alpha\beta}\Lambda_i\,, \\ &[\Lambda_i, \Lambda_8] = 0\,, \qquad [\Lambda_\alpha, \Lambda_8] = -\Lambda_{\alpha'}\,, \qquad \{\Lambda_{\alpha,\alpha'}\} = \frac{1}{4}\varepsilon_{\alpha\beta}\Lambda_8\,, \\ &\{\Lambda_{\alpha'}, \Lambda_{\beta'}\} = -\frac{1}{2}(\varepsilon\sigma_i)_{\alpha\beta}\Lambda_i\,, \qquad [\Lambda_{\alpha'}, \Lambda_8] = -\Lambda_\alpha\,, \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{1.6}$$

These relations show in particular that the additional three generators form a triplet under $osp(2,1)$.

Conventional Lie algebras like that of $su(2)$ have a $*$ or an adjoint operation † defined on them. For Λ_i, it is just $\Lambda_i^\dagger = \Lambda_i$. This follows from the fact that $a_i^\dagger$ is the adjoint of a_i. For $osp(2,1)$ and $osp(2,2)$, † is replaced by the grade adjoint ‡. On the oscillators, ‡ is defined by

$$a_i^\ddagger = a_i^\dagger\,, \quad (a_i^\dagger)^\ddagger = (a_i^\dagger)^\dagger = a_i\,, \qquad b^\ddagger = b^\dagger\,, \quad (b^\ddagger)^\ddagger = -b\,.$$

Hence $\ddagger = \dagger$ on bosonic oscillators.

On products of operators, $\ddagger$ is defined as follows. We assign the grade 0 to a_i, $a_j^\dagger$ and their products and 1 to b and $b^\dagger$. The grades are additive (mod 2). The grade of an operator L with definite grade is denoted by $|L|$. Then if L, M have definite grades, $(LM)^\ddagger \equiv (-1)^{|L||M|} M^\ddagger L^\ddagger$. Hence $(b^\dagger b)^\ddagger = b^\dagger b$ and

$$\Lambda_i^\ddagger = \Lambda_i\,, \qquad \Lambda_\alpha^\ddagger = -\varepsilon_{\alpha\beta}\Lambda_\beta\,, \qquad \Lambda_{\alpha'}^\ddagger = \varepsilon_{\alpha\beta}\Lambda_{\beta'} \qquad \Lambda_8^\ddagger = \Lambda_8\,. \tag{1.7}$$

1.3 Irreducible representations

Let $osp(2,0)$ denote $su(2)$, the Lie algebra of SU(2). Its IRR's are Γ_J^0, $J \in \mathbb{N}/2$. (Here $\mathbb{N} = \{0, 1, 2, \ldots\}$.) J has the meaning of angular momentum.

The $osp(2,1)$ algebra is of rank 1 just as $osp(2,0)$. We can take Λ_3 to be the generator of its Cartan subalgebra. Since

$$[\Lambda_3, \Lambda_4] = \frac{1}{2}\Lambda_4\,, \qquad [\Lambda_3, \Lambda_+ = \Lambda_1 + i\Lambda_2] = \Lambda_+\,,$$

Λ_4, Λ_+ are its raising operators. They commute:

$$[\Lambda_4, \Lambda_+] = 0\,.$$

In an IRR, both vanish on the highest weight vector. The eigenvalue $J \in \mathbb{N}/2$ of Λ_3 on the highest weight vector can be used to label its IRR's. They are denoted by Γ_J^1 in this paper.

When restricted to its subalgebra $osp(2,0)$, Γ_J^1 splits as follows:

$$\Gamma_J^1\Big|_{osp(2,0)} = \Gamma_J^0 \oplus \Gamma_{J-\frac{1}{2}}^0\,, \qquad J \geq \frac{1}{2}\,. \tag{1.8}$$

Γ_0^1 is the trivial IRR.

The dimension of Γ_J^1 is $4J+1$.

The graded Lie algebra $osp(2,2)$ is of rank 2. A basis for its Cartan subalgebra is $\{\Lambda_3\,, \Lambda_8\}$. Since

$$[\Lambda_3, \Lambda_4 + \Lambda_{4'}] = \frac{1}{2}(\Lambda_4 + \Lambda_{4'})\,, \qquad [\Lambda_8, \Lambda_4 + \Lambda_{4'}] = \Lambda_4 + \Lambda_{4'}\,,$$

$\Lambda_4 + \Lambda_{4'}$ serves as the raising operator for both Λ_3 and Λ_8. We also have that $\Lambda_1 + i\Lambda_2 = \Lambda_+$ is the raising operator for Λ_3 alone:

$$[\Lambda_3, \Lambda_+] = \Lambda_+\,, \qquad [\Lambda_8, \Lambda_+] = 0\,.$$

The raising operators $\Lambda_4 + \Lambda_{4'}$ and Λ_+ commute:

$$[\Lambda_4 + \Lambda_{4'}, \Lambda_+] = 0\,.$$

Both vanish on the highest weight vector in an IRR while the eigenvalues $J \in \mathbb{N}/2$ and $k \in \mathbb{Z}$ of Λ_3 and Λ_8 on the highest weight vector can be used as labels of the IRR. They are denoted in this paper by $\Gamma_J^2(k)$.

JHEP12(2005)002

The $osp(2,2)$ IRR's fall into two classes, the *typical* and *atypical* (or *short*) IRR's. In the typical IRR's, $|k| \neq 2J$ or $k = J = 0$, while in the atypical IRR's, $|k| = 2J \neq 0$. The typical IRR with $|k| \neq 2J$ restricted to $osp(2,1)$ splits as follows:

$$\Gamma^2_J(k)\big|_{osp(2,1)} = \Gamma^1_J \oplus \Gamma^1_{J-\frac{1}{2}}\,, \qquad J \geq \frac{1}{2}\,.$$

$\Gamma^2_0(0)$ is the trivial representation.

The atypical IRR's $\Gamma^2_J(\pm\frac{J}{2})$ $(J \geq 1/2)$ remain irreducible on restriction to $osp(2,1)$:

$$\Gamma^2_J\Big(\pm\frac{J}{2}\Big)\Big|_{osp(2,1)} = \Gamma^1_J\,.$$

$\Gamma^2_J(\pm J/2)$ can also be abbreviated to $\Gamma^2_{J\pm}$:

$$\Gamma^2_J\Big(\pm\frac{J}{2}\Big) \equiv \Gamma^2_{J\pm}\,, \qquad J \geq \frac{1}{2}\,.$$

$osp(2,2)$ admits the automorphism

$$\tau:\ \Lambda_i \to \Lambda_i\,, \qquad \Lambda_\alpha \to \Lambda_\alpha\,, \qquad \Lambda_{\alpha'} \to -\Lambda_{\alpha'}\,, \qquad \Lambda_8 \to -\Lambda_8 \tag{1.9}$$

which interchanges $\Gamma^2_J(\pm k)$:

$$\tau:\ \Gamma^2_J(k) \to \Gamma^2_J(-k)\,.$$

1.4 Casimir operators

The $osp(2,0) := su(2)$ Casimir operator K_0 is well-known:

$$K_0 = \Lambda_i^2\,.$$

The $osp(2,1)$ Casimir operator is

$$K_1 = \Lambda_i^2 + \varepsilon_{\alpha\beta}\Lambda_\alpha\Lambda_\beta\,.$$

We have that

$$K_1|_{\Gamma^1_J} = J(J+\frac{1}{2})\mathbb{1}\,.$$

The $osp(2,2)$ quadratic Casimir operator is

$$K_2 = K_1 - \varepsilon_{\alpha\beta}\Lambda_{\alpha'}\Lambda_{\beta'} - \frac{1}{4}\Lambda_8^2 := K_1 - V_0\,. \tag{1.10}$$

It has the property

$$\begin{aligned} K_2|_{\Gamma^2_J(k)} &= J^2 - \frac{k^2}{4}\,,\\ K_2|_{\Gamma^2_{J\pm}} &= 0\,. \end{aligned} \tag{1.11}$$

As already mentioned, the IRR's $\Gamma^2_{J\pm}$ can be distinguished by the sign of Λ_8 on the highest weight vector.

$osp(2,2)$ also has a cubic Casimir operator [7], but we will not have occasion to use it.

JHEP12(2005)002

1.5 Tensor products

The basic Clebsh-Gordan series we need to know is as follows:

$$\Gamma^1_J \otimes \Gamma^1_K = \Gamma^1_{J+K} \oplus \Gamma^1_{J+K-1/2} \oplus \cdots \oplus \Gamma^1_{|J-K|}\,.$$

1.6 The supertrace and the grade adjoint

Because of the decomposition (1.8), the vector space $\mathbb{C}^{4J+1}$ on which Γ^1_J acts can be written as $\mathbb{C}^{2J+1} \oplus \mathbb{C}^{2J}$ where the first term has angular momentum J and the second term has angular momentum $J - 1/2$. By definition, the first term is the even subspace and the second term is the odd subspace. The supertrace str of a matrix

$$M = \begin{pmatrix} P_{(2J+1)\times(2J+1)} & Q_{(2J+1)\times 2J} \\ R_{2J\times(2J+1)} & S_{2J\times 2J} \end{pmatrix}$$

is accordingly

$$\operatorname{str} M = trP - trS\,.$$

The grade adjoint $M^\ddagger$ can be calculated using the rules of graded vector spaces [8]. The result is

$$M^\ddagger = \begin{pmatrix} P^\dagger & -R^\dagger \\ Q^\dagger & S^\dagger \end{pmatrix}.$$

This formula is coherent with (1.7).

If $Q, R = 0$, we say that M is even, while if $P, S = 0$, we say that M is odd. We assign a number $|M| = 0, 1$ (mod 2) to even and odd matrices M respectively.

1.7 The free action

The space with $N = n$ has maximum angular momentum $J = n/2$. It carries the $osp(2,1)$ IRR $\Gamma^1_{n/2}$ which splits under $su(2)$ into $\Gamma^0_{n/2} \oplus \Gamma^0_{(n-1)/2}$. It carries either of the short $osp(2,2)$ IRR's as well.

The dimension of the Hilbert space with $N = n$ is $2n + 1$. We denote it by $\mathcal{H}^{2n+1}$. It is the direct sum $\mathcal{H}^{n+1} \oplus \mathcal{H}^n$ where $\mathcal{H}^{n+1}$ is the even subspace carrying the IRR $\Gamma^0_{n/2}$ and $\mathcal{H}^n$ is the odd subspace carrying the representation $\Gamma^0_{(n-1)/2}$. A basis for $\mathcal{H}^{2n+1}$ is

$$\frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} (b^\dagger)^{n_3} |0\rangle\,, \qquad \sum n_i = n\,, \qquad n_3 \in (0,1)\,, \quad (b^\dagger)^0 := \mathbb{1}\,. \tag{1.12}$$

The fuzzy SUSY $S_F^{2,2}$ (in the zero instanton sector) is the matrix algebra $Mat(4J+1) = Mat(2n+1)$. Just as S_F^2, it can be realized using oscillators. In terms of oscillators, a typical element is

$$\sum_{i,j} c^m_{i,j} (a_i^\dagger)^m (a_j)^m \; + \sum_{i,j} d^{m-1}_{i,j} (a_i^\dagger)^{m-1} (a_j)^{m-1} b^\dagger b\,, \qquad c^m_{i,j}, d^{m-1}_{i,j} \in \mathbb{C}\,.$$

It is to be restricted to the space $\mathcal{H}^{2n+1}$.

JHEP12(2005)002

The left- and right-actions

$$\Lambda_\rho^L M = \Lambda_\rho M\,, \qquad \Lambda_\rho^R M = (-1)^{|\Lambda_\rho||M|} M \Lambda_\rho$$

of $osp(2,\mathcal{N})$ on $Mat(2n+1)$ give two commuting IRR's of $osp(2,\mathcal{N})$. Here, $\Lambda_\rho \in osp(2,\mathcal{N})$, $\mathcal{N}=1,2$, $M \in Mat(2n+1)$ and both Λ_ρ and M are of definite grade $|\Lambda_\rho|, |M|$ (mod 2).

Combining the left- and right- representations, we get the grade adjoint representation

$$\mathrm{gad}\ :\ \Lambda_\rho \to \mathrm{gad}\,\Lambda_\rho = \Lambda_\rho^L - \Lambda_\rho^R\,, \qquad \rho \in (i,\alpha,\alpha',8)$$

of $osp(2,\mathcal{N})$.

With regard to gad , $Mat(4J+1)$ transforms under OSp(2,1) as

$$\Gamma_J^1 \otimes \Gamma_J^1 = \Gamma_{2J}^1 \oplus \Gamma_{2J-1/2}^1 \oplus \Gamma_{2J-1}^1 \oplus \cdots \oplus \Gamma_0^1\,. \tag{1.13}$$

$osp(2,2)$ acts on $Mat(4J+1)$ by L, R and gad representations as well. The L and R are the short representations $\Gamma_{J\pm}^2$ so that under gad , $Mat(4J+1)$ transforms as $\Gamma_{J+}^2 \otimes \Gamma_{J-}^2$. Its reduction can be inferred from (1.13) once we know that $\Gamma_j^2(0)|_{osp(2,1)} = \Gamma_j^1 \oplus \Gamma_{j-1/2}^1$. We will see this later. Hence

$$\Gamma_{J+}^2 \otimes \Gamma_{J-}^2 = \Gamma_{2J}^2(0) \oplus \Gamma_{2J-1}^2(0) \oplus \cdots \oplus \Gamma_0^2(0)\,.$$

The fuzzy field Φ is an element of fuzzy SUSY. The free action for Φ is

$$S_0 = \frac{f^2}{2}\,\mathrm{str}\,\Phi^\ddagger V_0 \Phi\,,$$

where f is a real constant and V_0 is an $osp(2,1)$-invariant operator. When restricted to the odd subspace, it becomes the Dirac operator of [11, 12].

The limit of this operator for $J=\infty$ was found by Fronsdal [9] and later used effectively by Grosse *et al.* [10] For $J=\infty$, it is the difference $K_1 - K_2$ of the Casimir operators K_1 and K_2 written as graded differential operators. This operator, for finite J, becomes

$$V_0 = \varepsilon_{\alpha\beta}(\Lambda_{\alpha'})(\Lambda_{\beta'}) + \frac{1}{4}(\Lambda_8)^2\,. \tag{1.14}$$

The simplifications of S_0 for this choice of V_0 is given elsewhere [6].

It is evident that V_0 is $osp(2,1)$-invariant. But it is less obvious that $\mathrm{gad}\,\Lambda_{\alpha'}$, $\mathrm{gad}\,\Lambda_8$ *anti*-commute with V_0:

$$\{\mathrm{gad}\,\Lambda_{\alpha'}, V_0\} = \{\mathrm{gad}\,\Lambda_8, V_0\} = 0\,. \tag{1.15}$$

This means that these generators are realized as *chiral* symmetries. Of these, $\mathrm{gad}\,\Lambda_8$, restricted to the odd sector, is just standard chirality. Thus, these generators, associated with $osp(2,2)/osp(2,1)$ are SUSY generalizations of conventional chirality.

We now show these results.

2. SUSY chirality

Let us first exhibit the highest weight vectors of the $su(2)$ IRR's which occur in $\Gamma^2_j(0)$. Here j is an integer. Referring to (1.8), we have that $\Gamma^1_j|_{su(2)} = \Gamma^0_j \oplus \Gamma^0_{j-1/2}$ for $j \geq 1$. The highest weight vector of Γ^0_j is $(a_1^\dagger a_2)^j$ as it commutes with Λ_4 and carries the eigenvalue j of $\mathrm{gad}\,\Lambda_3$. $\mathrm{gad}\,\Lambda_5$ maps it to $-j(a_1^\dagger a_2)^{j-1}\Lambda_4$, the highest weight vector of $\Gamma^0_{j-1/2} \subset \Gamma^1_j$. Thus

$$\begin{array}{rccccc} \Gamma^1_j\Big|_{su(2)} & = & \Gamma^0_j & \oplus & \Gamma^0_{j-1/2} & \\ \left.\begin{array}{c}\text{Highest weight}\\ \text{vectors}\end{array}\right\} & & (a_1^\dagger a_2)^j & \xrightarrow{\mathrm{gad}\Lambda_5} & -j(a_1^\dagger a_2)^{j-1}\Lambda_4\,, & j \geq 1\,. \end{array} \tag{2.1}$$

The equation also indicates the operator mapping one highest weight vector of $su(2)$ to the other.

Next consider $\Gamma^1_{j-1/2} \supset \Gamma^0_{j-1/2} \oplus \Gamma^0_{j-1}$ for $j \geq 1$. To distinguish the $su(2)$ IRR's here from those in Γ^1_j, we put a prime on them:

$$\Gamma^1_{j-1/2}\Big|_{su(2)} = \Gamma^{0'}_{j-1/2} \oplus \Gamma^{0'}_{j-1}\,.$$

The highest weight state of $\Gamma^1_{j-1/2}$, commuting with Λ_4 and with eigenvalue $j-1/2$ for $\mathrm{gad}\,\Lambda_3$ is $-j(a_1^\dagger a_2)^{j-1}\Lambda_6$. And Λ_5 maps it to the highest weight vector X_{j-1} of $\Gamma^{0'}_{j-1}$. We show X_{j-1} below. Thus

$$\begin{array}{rcccc} \Gamma^1_{j-1/2}\Big|_{su(2)} & = & \Gamma^{0'}_{j-1/2} & \oplus & \Gamma^{0'}_{j-1} \\ \left.\begin{array}{c}\text{Highest weight}\\ \text{vectors}\end{array}\right\} & & -j(a_1^\dagger a_2)^{j-1}\Lambda_6 & \xrightarrow{\mathrm{gad}\Lambda_5} & X_{j-1}\,, \end{array} \tag{2.2}$$

$$X_{j-1} = \frac{j-2J-1}{4}(a_1^\dagger a_2)^{j-1} + \frac{1-2j}{4}(a_1^\dagger a_2)^{j-1}b^\dagger b\,, \qquad j \geq 1\,.$$

In calculating X_{j-1}, we use

$$\Lambda_4\Lambda_6 = -\frac{1}{4}(a_1^\dagger a_2)(2b^\dagger b - 1)\,, \qquad a^\dagger a + b^\dagger b = 2J\,.$$

Now $\mathrm{gad}\,\Lambda_7$, $\mathrm{gad}\,\Lambda_8$ map the vectors in (2.1) to the vectors in (2.2). The full table is

$$\begin{array}{cccccccc} & & \Gamma^1_j & \ni & (a_1^\dagger a_2)^j & \xrightarrow{\mathrm{gad}\Lambda_5} & -j(a_1^\dagger a_2)^{j-1}\Lambda_4 & \\ & \nearrow & & & \in \Gamma^0_j & & \in \Gamma^0_{j-1/2} & \\ \Gamma^2_j(0) & & & & \mathrm{gad}\,\Lambda_7 \downarrow & \swarrow \mathrm{gad}\,\Lambda_8 & \downarrow \mathrm{gad}\,\Lambda_7 & \\ & \searrow & & & & & & \\ & & \Gamma^1_{j-1/2} & \ni & -j(a_1^\dagger a_2)^{j-1}\Lambda_6 & \xrightarrow{\mathrm{gad}\Lambda_5} & X_{j-1}\quad , & j \geq 1\,. \\ & & & & \in \Gamma^{0'}_{j-1/2} & & \in \Gamma^{0'}_{j-1} & \end{array} \tag{2.3}$$

For $j = 0$, we get the trivial IRR of $osp(2, \mathcal{N})$'s.

JHEP12(2005)002

Eq. (2.3) shows that $\operatorname{gad}\Lambda_{\alpha'}$, $\operatorname{gad}\Lambda_8$ map the vectors of Γ^1_j to those of $\Gamma^1_{j-1/2}$ $(j \geq 1)$ and vice versa. So if V_0 has opposite eigenvalues in the representations in (2.3), then we can conclude that[1]

$$\{\operatorname{gad}\Lambda_{\alpha'}, V_0\} = \{\operatorname{gad}\Lambda_8, V_0\} = 0$$

identically, since $V_0|_{\Gamma^1_0} = 0$. That means that these operators associated with $osp(2,2)/osp(2,1)$ are *chirally* realized symmetries.

3. Eigenvalues of V_0

As V_0 is an $osp(2,1)$ scalar, it is enough to compute its eigenvalue on the highest weight state of Γ^1_j to find $V_0|_{\Gamma^1_j}$.

As $\Lambda_{4'} = \Lambda_6$ commutes with $(a_1^\dagger a_2)^j$, we have that

$$\varepsilon_{\alpha\beta}\operatorname{gad}\Lambda_{\alpha'}\operatorname{gad}\Lambda_{\beta'}(a_1^\dagger a_2)^j = (\operatorname{gad}\Lambda_{4'}\operatorname{gad}\Lambda_{5'} + \operatorname{gad}\Lambda_{5'}\operatorname{gad}\Lambda_{4'})(a_1^\dagger a_2)^j$$

where the sign of the second term has been switched as it is zero anyway. Thus the left-hand side of the previous formula can be written as

$$\operatorname{gad}\{\Lambda_{4'}, \Lambda_{5'}\}(a_1^\dagger a_2)^j = \frac{1}{2}\operatorname{gad}\Lambda_3(a_1^\dagger a_2)^j = \frac{j}{2}(a_1^\dagger a_2)^j\,.$$

Also

$$\operatorname{gad}\Lambda_8(a_1^\dagger a_2)^j = 0\,.$$

Hence

$$V_0(a_1^\dagger a_2)^j = \frac{j}{2}(a_1^\dagger a_2)^j\,. \tag{3.1}$$

One quick way to evaluate $V_0|_{\Gamma^1_{j-1/2}}$ is as follows. Since $K_1|_{\Gamma^1_j} = j(j+1/2)$, we have

$$K_2|_{\Gamma^1_j} = (K_1 - V_0)|_{\Gamma^1_j} = j^2\,. \tag{3.2}$$

But K_2 is $osp(2,2)$-invariant. Hence

$$K_2|_{\Gamma^1_{j-1/2}} = j^2\,.$$

Since also $K_1|_{\Gamma^1_{j-1/2}} = j(j-1/2)$, we have

$$V_0|_{\Gamma^1_{j-1/2}} = (K_1 - K_2)|_{\Gamma^1_{j-1/2}} = -\frac{j}{2}\mathbb{1}\,.$$

Thus V_0 has opposite eigenvalues on Γ^1_j and $\Gamma^1_{j-1/2}$.

It is important to notice that

$$K_2 = (2V_0)^2\,.$$

That is, $2V_0$ is a square root of K_2, a bit in the way that the Dirac operator is the square root of the laplacian.

[1] To show that $\{\operatorname{gad}\Lambda_6, V_0\} = 0$ we use the fact that $\operatorname{gad}\Lambda_6 = -[\operatorname{gad}\Lambda_4, \operatorname{gad}\Lambda_8]$. The result follows from the graded Jacobi identity.

4. Fuzzy SUSY instantons

The manifold S^2 admits twisted U(1) bundles labelled by a topological index or Chern number $k \in \mathbb{Z}$. In the algebraic language, sections of vector bundles associated with these U(1) bundles are described by elements of projective modules [14, 15].

When S^2 becomes the graded supersphere $S^{2,2}$, we expect these modules to persist, and become in some sense supersymmetric projective modules. That is in fact the case. We shall see that explicitly after first studying their fuzzy analogues.

The projective modules on S^2 and S^2_F are associated with $\mathrm{SU}(2) \simeq S^3$ via Hopf fibration and Lens spaces. In the same way, the supersymmetric projective modules on $S^{2,2}$ and $S_F^{2,2}$ get associated with $osp(2,1)$ and $osp(2,2)$.

The fuzzy algebra $S_F^{2,2}$ of previous sections is to be assigned $k = 0$. The elements of this algebra are square matrices mapping the space with $N = 2J$ to the same space $N = 2J$. We emphasize the value of k by writing $S_F^{2,2}(0)$ for $S_F^{2,2}$. $S_F^{2,2}(0)$ is a bimodule for $osp(2,2)$ as the latter can act on the left or right of $S_F^{2,2}(0)$ by the IRR's $\Gamma^2_{J\pm}(0)$.

For $k \neq 0$, $S_F^{2,2}(k)$ is not an algebra. It can be described using projectors [15, 12] or equally well as maps of the vector space with $N = 2J$ to the one with $N = 2J + k$ [10]. (We take $J + \frac{k}{2} \geq 0$. If $k < 0$, this means $J \geq |k|/2$.) If a basis is chosen for domain and range of $S_F^{2,2}(k)$, their elements become rectangular matrices with $2J + k$ rows and $2J$ columns. $S_F^{2,2}(k)$ as well is a bimodule for $osp(2,2)$. The latter acts by $\Gamma^2_{(J+\frac{k}{2})+}$ on the left of $S_F^{2,2}(k)$ and by Γ^2_{J-} on the right of $S_F^{2,2}(k)$.

The invariant associated with $S_F^{2,2}(k)$ is just k. The meaning of k is

$$k = \text{Dimension of range of } S_F^{2,2}(k) - \text{Dimension of domain of } S_F^{2,2}(k)\,.$$

Scalar fields Φ are now elements of $S_F^{2,2}(k)$ while V_0 is replaced by a new operator V_k which incorporates the appropriate connection and "topological" data. We now argue, using index theory and other considerations, that the $osp(2,1)$-invariant V_k is fixed by the requirement

$$V_k^2 = K_2$$

where K_2 is the Casimir invariant for $\Gamma^2_{(J+k)+} \otimes \Gamma^2_{J-}$.

5. Fuzzy SUSY zero modes and their index theory

We begin by analyzing the $osp(2,1)$ and $osp(2,2)$ representation content of $S_F^{2,2}(k)$.

As regards the gad representation of $osp(2,1)$, it transforms according to

$$\Gamma^1_{J+k}\otimes\Gamma^1_J = \left(\Gamma^1_{2J+\frac{k}{2}} \oplus \Gamma^1_{2J+\frac{k}{2}-\frac{1}{2}}\right) \oplus \left(\Gamma^1_{2J+\frac{k}{2}-1} \oplus \Gamma^1_{2J+\frac{k}{2}-\frac{3}{2}}\right) \oplus \cdots \oplus \left(\Gamma^1_{\frac{|k|}{2}+1} \oplus \Gamma^1_{\frac{|k|}{2}+\frac{1}{2}}\right) \oplus \Gamma^1_{\frac{|k|}{2}}\,.$$

The analogue of (2.3) is:

$$2J + \frac{k}{2} \geq j \geq \frac{|k|}{2} + 1\,,$$

$$
\begin{array}{ccl}
 & & \Gamma^1_j \;\ni\; \Gamma^0_j \;\oplus \Gamma^0_{j-1/2} \\
 & \nearrow & \\
\Gamma^2_j(k) & & \\
 & \searrow & \\
 & & \Gamma^1_{j-1/2} \ni \Gamma^0_{j-1/2} \oplus \Gamma^0_{j-1}\,.
\end{array}
\tag{5.1}
$$

Here $|k| \geq 1$. For $j = |k|/2$, we get the atypical representation of $osp(2,2)$:

$$\Gamma^2_{\frac{|k|}{2}}(k) \to \Gamma^1_{\frac{|k|}{2}} = \Gamma^0_{\frac{|k|}{2}} \oplus \Gamma^0_{\frac{|k|}{2}-1}\,.$$

All this becomes explicit during the following calculation of the eigenvalues of K_2.

5.1 Spectrum of K_2

For $k > 0$, the highest weight vector with angular momentum

$$j = m + \frac{|k|}{2}\,, \qquad m = 0, 1, \ldots$$

is

$$(a_1^\dagger)^{|k|}(a_1^\dagger a_2)^m\,.$$

Since

$$\mathrm{gad}\,\Lambda_8 (a_1^\dagger)^{|k|}(a_1^\dagger a_2)^m = |k|(a_1^\dagger)^{|k|}(a_1^\dagger a_2)^m\,,$$

it is the highest weight vector of $\Gamma^2_j(|k|)$. Thus $\Gamma^2_j(|k|)$ occurs in the reduction of the $osp(2,2)$ action on $S_F^{2,2}(|k|)$.

General theory [7] tells us the branching rules of $\Gamma^2_j(|k|)$ as in (5.1). This equation is thus established for $k > 0$.

We can check as before that

$$\varepsilon_{\alpha\beta}\,\mathrm{gad}\,\Lambda_{\alpha'}\,\mathrm{gad}\,\Lambda_{\beta'}\,(a_1^\dagger)^{|k|}(a_1^\dagger a_2)^m = \frac{1}{2}\left(m + \frac{|k|}{2}\right)(a_1^\dagger)^{|k|}(a_1^\dagger a_2)^m$$

while

$$\frac{1}{4}(\mathrm{gad}\,\Lambda_8)^2 (a_1^\dagger)^{|k|}(a_1^\dagger a_2)^m = \frac{k^2}{4}$$

and

$$K_1|_{\Gamma^1_j} = j(j+1)\mathbb{1}\,.$$

We thus have [7]

$$K_2|_{\Gamma^2_j(|k|)} = \left(j^2 - \frac{k^2}{4}\right)\mathbb{1}\,.$$

For $k < 0$,

$$(a_2)^{|k|}(a_1^\dagger a_2)^m$$

is the highest weight vector for angular momentum

$$j = m + \frac{|k|}{2}\,.$$

Since

$$\mathrm{gad}\,\Lambda_8 (a_2)^{|k|}(a_1^\dagger a_2)^m = -|k|(a_2)^{|k|}(a_1^\dagger a_2)^m\,,$$

it is the highest weight vector of $\Gamma_j^2(-|k|)$. Hence $\Gamma_j^2(-|k|)$ occurs in the reduction of the $osp(2,2)$ action on $S_F^{2,2}(-|k|)$. We thus establish (5.1) for $k<0$ as well.

The eigenvalues of V_k, when restricted to Γ_j^1 and $\Gamma_{j-1/2}^1$ and for $j \geq \frac{|k|}{2}+1$, are $\pm\sqrt{j^2-\frac{k^2}{4}}$. These eigenvalues are not zero. Hence the $osp(2,2)$ operators which intertwine these representations, mapping vectors of one representation to the other, *anticommute* with V_k: they are *chiral* symmetries for these representations. For $j=\frac{|k|}{2}$, V_k vanishes while the representation space carries the atypical representation $\Gamma^2_{\frac{|k|}{2}}(k)$ of $osp(2,2)$. Hence we can say that the above chiral operators all anticommute with $V_k|_{J=0}$. Hence these operators anticommute with V_k (for any j, on all vectors of $S_F^{2,2}(k)$) just as standard chirality anticommutes with the massless Dirac operator.

For $k=0$, these operators were $\Lambda_{\alpha'}$, Λ_8. But they change with k. They can be worked out. They do not occur in subsequent discussion and hence we do not show them here.

We now establish that V_k is the correct choice of the action for the fuzzy SUSY action S_k:

$$S_k = \text{const. str}\,\Phi^\ddagger\, V_k\, \Phi\,. \tag{5.2}$$

This formula is valid also for $k=0$ as we saw earlier. We here focus on $k\neq 0$.

The Dirac operators D for fuzzy spheres of instanton number k are known [12]. We first show that V_k coincides with this operator on the Dirac sector.

It is enough to focus on typical $osp(2,2)$ IRR's since both the Dirac operator and V_k vanish on the grade-odd sector of $\Gamma^1_{\frac{|k|}{2}}$. Thus consider $\Gamma_j^2(k)$ for $j\geq \frac{1}{2}|k|+1$. Angular momentum J in the Dirac sector of $\Gamma_j^2(k)$ is $j-1/2$. Hence

$$V_k^2\big|_{\Gamma_j^2(k)\ \text{Dirac sector}} = \left(J-\frac{|k|-1}{2}\right)\left(J+\frac{|k|+1}{2}\right)\mathbb{1}\,.$$

qSubstituting $J=n+\frac{|k|-1}{2}$ and identifying $|k|=2T$, we get the answer of [12]:

$$V_k^2\big|_{\Gamma_j^2(k)\text{Dirac sector}} = n(n+2T)\mathbb{1}\,.$$

Hence $V_k^2\big|_{\Gamma_j^2(k)\ \text{Dirac sector}}$ is the correct Dirac operator .

This result and the $osp(2,1)$-invariance of V_k are compelling reasons to identify it as the SUSY generalization of the Dirac and laplacian operators for $k\neq 0$.

5.2 Index theory and zero modes

There is also further evidence supporting the correctness of V_k: It gives the SUSY generalization of index theory.

Thus one knows that 1) the Dirac operator has $|k|$ zero modes for instanton number k on S^2 and on the fuzzy sphere $S_F^2(k)$, [12, 13] and 2) they are left- (right-) chiral if $k>0$ ($k<0$), 3) charge conjugation interchanges these chiralities.

More precisely if $n_{L,R}$ are the number of left- and right-chiral zero modes,

$$n_L - n_R = k\,.$$

This number is "topologically stable". The meaning of this statement in the fuzzy case can be found in [12].

If the Dirac operator is SU(2)-invariant, these zero modes organize themselves into SU(2) multiplets with angular momentum $\frac{|k|}{2}$ [12, 13].

Now V_k has zero modes which form the atypical multiplet $\Gamma^2_{\frac{|k|}{2}}(k)$ of $osp(2,2)$. The number of zero modes is $2|k|+1$. Of these, $|k|$ correspond to the grade odd sector and can be identified with the zero modes of S^2 and $S^2_F(k)$ Dirac operators. The remaining grade even $(|k|+1)$ zero modes are their SUSY-partners.

The zero modes transform by inequivalent IRR's of $osp(2,2)$ for the two signs of k. These two atypical $osp(2,2)$ representations are SUSY generalizations of left- and right-chiralities.

Identifying charge conjugation with the automorphism (1.9), we see that it exchanges these two IRR's just as it exchanges chiralities in the Dirac sector.

6. Final remarks

In this paper, we have extended the work of [6] on the fuzzy SUSY model on S^2 to the instanton sector. A SUSY generalization of index theory and zero modes of the Dirac operator has also been established.

Following [6], we can try introducing interactions involving just Φ. For $k \neq 0$, Φ can be thought of as a rectangular matrix. So $\Phi^\ddagger\Phi$ and $\Phi\Phi^\ddagger$ are square matrices of different sizes acting on $osp(2,2)$ representations with $N = n$ and $N = n+k$. A typical interaction may then be

$$\lambda\, \mathrm{str}(\Phi^\ddagger\Phi)^2 \tag{6.1}$$

where str is over the space with $N = n$, the domain of $\Phi^\ddagger\Phi$. (But note that (6.1) and the use of str in interactions require further study.)

Fuzzy SUSY gauge theories remain to be formulated. The investigation of the graded commutative limit $n \to \infty$ with k fixed has also not been done for $k \neq 0$.

Numerical simulations on fuzzy SUSY models are being initiated.

Acknowledgments

We are part of a collaboration on fuzzy physics and have benefited by inputs from its members. We are particulary grateful to Marco Panero for carefully reading the paper and suggesting several corrections. This work was supported in part by the DOE grant DE-FG02-85ER40231 and by NSF under contract number INT9908763.

References

[1] G. Landi, *Projective modules of finite type over the supersphere* $S^{2,2}$, *Differ. Geom. Appl.* **14** (2001) 95 [math-ph/9907020].

[2] K. Hasebe and Y. Kimura, *Fuzzy supersphere and supermonopole*, *Nucl. Phys.* **B 709** (2005) 94 [hep-th/0409230] and references therein.

[3] K. Fujikawa, *Path integral measure for gauge invariant fermion theories*, *Phys. Rev. Lett.* **42** (1979) 1195.

[4] K. Fujikawa, *Path integral for gauge theories with fermions*, *Phys. Rev.* **D 21** (1980) 2848, erratum *ibid.* **D22** (1980) 1499.

[5] A.P. Balachandran, G. Marmo, V.P. Nair and C.G. Trahern, *A nonperturbative proof of the nonabelian anomalies*, *Phys. Rev.* **D 25** (1982) 2713.

[6] A.P. Balachandran, B. Dolan, D. O'Connor, M. Panero and S. Kürkçüoğlu, in preparation.

[7] L. Frappat, P. Sorba and A. Sciarrino, *Dictionary on Lie superalgebras*, hep-th/9607161.

[8] M. Scheunert, W. Nahm and V. Rittenberg, *Graded Lie algebras: generalization of hermitian representations*, *J. Math. Phys.* **18** (1977) 146.

[9] C. Fronsdal, *Essays on supersymmetry*, C. Fronsdal and D. Reidel eds., Publishing Company, Dordrecht 1986.

[10] H. Grosse, C. Klimčík and P. Prešnajder, *Field theory on a supersymmetric lattice*, *Commun. Math. Phys.* **185** (1997) 155 [hep-th/9507074].

[11] H. Grosse and P. Prešnajder, *The dirac operator on the fuzzy sphere*, *Lett. Math. Phys.* **33** (1995) 171.

[12] A.P. Balachandran and G. Immirzi, *The fuzzy Ginsparg-Wilson algebra: a solution of the fermion doubling problem*, *Phys. Rev.* **D 68** (2003) 065023 [hep-th/0301242].

[13] A.P. Balachandran and S. Vaidya, *Instantons and chiral anomaly in fuzzy physics*, *Int. J. Mod. Phys.* **A 16** (2001) 17 [hep-th/9910129].

[14] J.M. Gracia-Bondia, J.C. Varilly and H. Figueroa, *Elements of noncommutative geometry*, Birkhaeuser, Boston 2001.

[15] S. Baez, A.P. Balachandran, B. Ydri and S. Vaidya, *Monopoles and solitons in fuzzy physics*, *Commun. Math. Phys.* **208** (2000) 787 [hep-th/9811169].

JHEP12(2005)002